Computing the Expansion History of the Universe

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Have you ever wondered how scientists determine the size or age of the universe? These bulk properties are a bit mysterious unless you can calculate them for yourself. The physical model of the expanding universe is the primary focus of our class and you'll be studying it in detail. This model relies on conservation of energy (kinetic and potential) and the thermodynamic properties of fluids and it is calculated in a coordinate system that expands with a scale factor, $a(t)$. To tie the model to reality, we have to understand how it impacts observable quantities like the redshift of a galaxy and the brightness or size of a galaxy. After all, it is only when we compare observations to a particular model that we find out whether the model is true or not.

In the first part of this project, we will use $a(t)$ to compute observable quantities. In the second part, we will use the general solution to the Friedman Equation to determine $a(t)$ for any universe of our choosing. This allows us to calculate the age and size of the universe. Then we'll explore the parameter space to determine how close Ryden's Benchmark model comes to the current best-measured parameters.

Part 1 – Observable Quantities

Telescopes generally point at astronomical sources to measure their photon intensity, spectra, and angular extent on the sky. At large distances these observables depend on the geometry and expansion rate of the universe. In fact, the expansion leads directly to an observed reddening of distant objects. We define this redshift in terms of the wavelength of light. If a distant galaxy emits light of a wavelength, λ_e, (*e* is for emitted) we will observe its redshifted wavelength, λ_o, (*o* is for observed) and the redshift is defined as (Ryden Eqn 2.4)

$$z = (\lambda_o - \lambda_e)/\lambda_e \tag{1}$$

The spectral lines of hydrogen, helium, and a number of other elements are routinely measured in undergraduate laboratories and you've probably seen this yourself when you studied optics. Remember that the each wavelength of light is a specific color. When the color changes, so does the wavelength. In astronomy, a spectrometer is used to measure the observed wavelength of astronomical objects with strong spectral lines. Since we already know the emitted wavelengths from our laboratory studies we can determine how the wavelength has changed and we call the relative change the redshift. A redshift can result from a Doppler shift due to the velocity of the astronomical source or from the expansion of the universe. In general, redshifts are a combination of the two. Beyond a redshift of about 0.03, however, the expansion of the universe dominates and the Doppler shift can be neglected.

To understand redshifts due to the expanding universe we need to see how length is defined during the expansion. The Robertson-Walker metric expresses the observed length, *ds*, in terms of the general relativistic space-time elements in spherical coordinates, *dt*, *dr*, and $d\Omega = \sqrt{d\theta^2 + sin^2\theta d\phi^2}$. (Ryden 3.25)

$$ds^2 = -c^2dt^2 + a(t)^2[dr^2 + S_K(r)^2 d\Omega^2] \tag{2}$$

where *c* is the speed of light, *a(t)* is a unitless scale factor that describes the spatial expansion of the metric, and $S_k(r)$ accounts for the curvature of space.

$$S_k(r) = \begin{cases} R_0 \sin(r/R_0) & \text{for } \kappa = +1 \\ r & \text{for } \kappa = 0 \\ R_0 \sinh(r/R_0) & \text{for } \kappa = -1 \end{cases} \tag{3}$$

where R_0 is the radius of curvature of the metric. Our universe appears to be flat with $S_k(r) = r$, but the metric allows for positive curvature, $\kappa = +1$, and negative curvature, $\kappa = -1$. Notice that the flat metric reduces to spherical coordinates with the additional special relativistic term, *-cdt*, and the expansion scale factor, *a(t)*. For convenience we set the scale factor to unity at the present time, $a(t_o) = 1$.

The curvature, R_0, and the sign of the curvature, κ, are determined from the Friedman equation: (Ryden 4.31)

$$\frac{\kappa}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1) \tag{4}$$

The curvature, R_0 is related to whether the total energy density is greater or less than the critical energy density ($\Omega_0 = \Omega_{m,0}+\Omega_{r,o}+\Omega_{\Lambda,0}$). This is one of the few times that we can determine two variables with one equation. For example, if $\Omega_0>1$, then the right-hand side is positive, κ=+1, and with this information we can solve for the radius of curvature, $R_0 = \frac{c}{H_0\sqrt{\Omega_0 - 1}}$.

Locations at fixed coordinates, r, θ, and ϕ in this metric are called comoving because they are observed to move in relation to each other by the scale factor, *a(t)*. We use light traveling between comoving emission and observation points to measure the comoving distance interval, *dr*. Let's set the origin of the coordinate system at the telescope that observes the light. In this coordinate system a photon travels radially toward the observation point at a constant angle, θ and ϕ, from the emitting source. This means that $d\Omega=0$. Furthermore, light travels along null geodesics defined by $ds = 0$ which allows us to solve for the comoving distance interval, $dr = c\, dt/a(t)$.

Two important results come from this. First, if we consider the wavelength of a photon $\lambda = cdt$ where dt is the period of the photon's oscillation, we find that this is $a(t_e)dr$ at the emission time and $a(t_o)dr$ when it is observed. For these comoving observers, dr is the same at both times and it is easy to show that the scale factor of the expanding universe, $a(t)$, is related to the redshift by: (Ryden Eqn 3.46)

$$z = \frac{1}{a(t_e)} - 1 \qquad (5)$$

where we have set today's scale factor to unity, $a(t_o) = 1$. By measuring the redshift, z, of an astronomical source as a shift in wavelength, we learn the value of the scale factor at the time the light was emitted.

Second, we find the line-of-sight distance to the source at the time we observe it. This is called the conformal distance and it is found by integrating over time from the observed time, t_o, backward to the emission time, t_e. (Ryden Eqns. 3.39 and 5.35)

$$D_c = \int_0^{D_c} dr = c \int_{t_o}^{t_e} \frac{dt}{a(t)} \qquad (6)$$

You may remember the importance of the proper distance in relativity. It is defined as the distance observed at a single time, t_i. When time is a constant, $dt = 0$ and the radial proper distance is $D_p(t_i) = \int ds = a(t_i) \int dr$. The proper distance today is the conformal distance $D_p(t_o)=D_c$. In an expanding universe the proper distance was smaller at the time the light was emitted $D_p(t_e)=a(t_e)D_c$.

Now that we have good definitions for distance, we can talk about what people see in their telescopes. The distance factors are derived in Ryden Ch. 7 for the flat universe where $S_k(D_p)=D_p$. Here we extend the discussion to include curvature. The observed width of a galaxy on the sky, $\Delta\Omega$, is related to the galaxy's diameter:

$$\text{true galaxy diameter} = D_A\, \Delta\Omega \qquad (7)$$

where D_A is the angular diameter distance. To get a feel for this, imagine that one night you look up and see the moon. If you extend your arm to point at one edge of the moon and then move it to the other edge of the moon, the angle that your arm moves is $\Delta\Omega$. A close look at the metric shows that D_A is simply defined by the coefficients of the $d\Omega$ term:

$$D_A = a(t_e) S_K(D_p) \qquad (8)$$

Back in geometry, the galaxy diameter was called the arc length and you may recognize that Eqn. 7 in flat polar coordinates becomes $ds = rd\theta$.

Similarly, the observed brightness of a source depends on how far away it is. Imagine that you are looking at headlights in the distance on a dark night. As the headlights get closer to you, you perceive them as brighter. The headlights don't change their luminosity, rather your observation of them changes. The observed brightness of a source is characterized by the flux of photons into the aperture of a telescope during the exposure time and has units of photons/(m^2 s). Since photons are emitted in all directions, the fraction that make it into a fixed aperture at a distance, r, goes as the inverse of a spherical surface area, $1/4\pi r^2$. The brightness also depends on the intrinsic luminosity of the source defined by the total number of emitted photons/second in all directions. The observed flux is related to the luminosity by the luminosity distance, D_L.

$$\text{measured flux} = \text{true luminosity}/(4\pi D_L^2) \tag{9}$$

The luminosity distance is constructed to make the equation look geometrical, but since the photons spread out over a spherical area related to $d\Omega$ and are also redshifted during transit, it depends both on the curvature of the universe and the redshift.

$$D_L = S_K(D_p)/a(t_e) \tag{10}$$

Finally, it is rare to measure a luminosity or flux directly. Usually we find ourselves integrating the flux and describing the brightness of a source by its magnitude. The absolute magnitude, M, of a source is given by: (Ryden Eqn. 7.48)

$$M = m - 5\log\left(\frac{D_L}{10pc}\right) \tag{11}$$

where m is the apparent magnitude and the second term is the distance modulus, DM = 5log(DL/ 10pc). Notice that the distance modulus depends only on the luminosity distance which can be computed directly from the metric at any emission time. A prediction of DM exists for every specific cosmological model of $S_k(r)$ and $a(t_e)$. Direct tests of the expansion have been made by measuring the apparent magnitude for sources with known absolute magnitude and comparing the difference, $m - M$, to the predicted distance modulus, DM. These tests lead to the 2011 Nobel Prize in Physics awarded to Saul Perlmutter, Brian Schmidt, and Adam Riess.

Key point: All of these measurable quantities can be computed if we can just figure out $a(t)$, R_0, and H_0.

There are a few cases where the scale factor can be computed analytically and in this part of the project, it's good to start with one of those. The solution for the Matter Only universe ($\Omega_0 = \Omega_{m,0} = 1$) is

$$H_0(t_e - t_o) = \frac{2}{3}(a^{\frac{3}{2}} - 1) \tag{12}$$

where H_0 is the Hubble constant. (note: H_0 is not a function of t_e-t_o, but rather, the left-hand side is $H_0 \times (t_e - t_o)$.) The function is plotted as a dotted line in Ryden's Figure 6.1 and we reproduce it in Figure 1 below. We will choose t_o=now, and measure t_e, as a time in the past or future. Right now, t_e=t_o, and the left-hand side is zero. What is $a($now) so that the right-hand side is also zero? Let's use $a(t)$ to understand redshift, and the distance factors.

General Instructions: If you wish to do this assignment without the step-by-step instructions, feel free to pick any computing language of your choice. Start by defining the parameters $H_0 = 70$ km/s/Mpc and $\Omega_0 = 1$, the constant, *c,* and conversions. Since we want to get about 4 significant figures of accuracy out of this computation, we need to use constants and conversions that are accurate to 6 significant figures as shown in Table 1 below. Next, make an array of log(a) from -6 to 0.5, incrementing in steps of 0.01 or so. Make additional arrays from the first to hold the values of a, z, $H_0(t_e\text{-}t_o)$. Debug the results using columns A-F in the spreadsheet shown in Figure 4 and by reproducing Figure 1. What is the age of the universe?

Next, integrate Eqn. 6 to find D_c being very careful to set the integration limits from t_o to t_e. You may use a trapezoid method, Simpson's method, or Romberg's method. Set $D_c = 0$ at t_o and then integrate backward to an emission time, t_e, in the past. Again, check that D_c is correct using the spreadsheet in Figure 4. Next, find κ and R_0 so that you can compute $S_k(D_c)$. To check positive curvature: set Ω_0= 1.05 and check that S_k = 8275.9522 Mpc where log(a) = -6. Then set Ω_0=0.95 and check that S_k= 8845.4898. Mpc where log(a) = -6. Finally, compute D_c/D_H, D_A/D_H, D_L/D_H, and DM. Recreate the plots in Figures 1 and 3. Skip ahead to page 10 where it says **Report**.

Excel Instructions: If you prefer more explanation and a detailed approach, here's how to do the computation in Excel.

A) Our first objective is to compute a(t). One way to do this is to invert Eqn. 12 to find an expression for $a(t_e)$. In later calculations, it won't be possible to do this, so we want to get good at using Eqn.12 as is. The time scales of interest to us extend from seconds to billions of years. To cover all the time scales of interest, start with log(a) instead of a. Take a look at the example spreadsheet in Figure 4 to see how the log(a) column should look. Create a spreadsheet column log(a). Compute a from the log(a) and then use Eqn. 12 to compute $H_0(t_e - t_0)$ from a. This should produce columns A, B, and C in your spreadsheet. Check them by recreating the dotted line in Figure 1 for yourself.

B) Column D in the spreadsheet shown in Figure 4 is emission time, t_e. It is easily computed using H_0 = 70 km/s/Mpc and the time right now, t_o = 0 seconds. For parameters, like H_0, you'll want to put them in a cell at the top and use them in equations. If you type them in all over the place, you'll have to debug the code every time you change their value. If you don't know how to anchor a number in

speed of light	2.99792×10^5 km/s
Seconds/year (including leap seconds)	3.15581×10^7
Mpc/km	3.24078×10^{-20}

Table 1: Parameters and conversions with 6 significant figures.

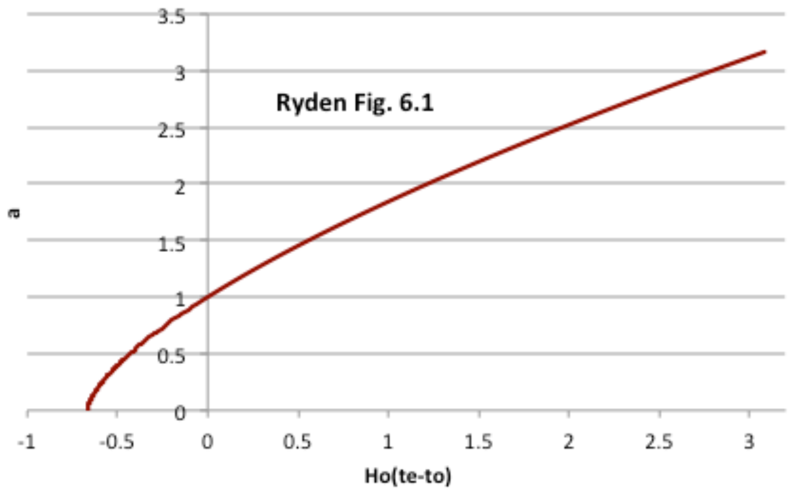


Figure 1: Recreation of the dotted line in the lower panel of Ryden Fig. 6.1 for the Matter Only universe.

an Excel equation, please get help. It's something every college student should learn to do. **Notice that the spreadsheet is color-coded**. Blue cells are computed with equations (they shouldn't have any numbers typed in by hand). Black cells are cells you have to enter by hand.

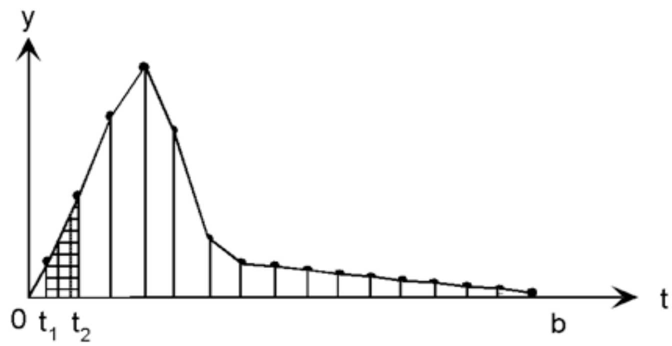
C) Column E is the age of the universe in years. Notice that the emission time is measured backward from now and that age is measured forward from the Big Bang. If we start the age at 0 years, what is the age today?

D) Compute the redshift column using Eqn. 5.

E) We will want to compute some more parameters and constants before tackling the distance factors. It's important to use constants and conversion factors that are accurate to 6 significant figures. See Table 1. Put them at the top of the spreadsheet with the cosmological parameters like H_0.

Go back and **fix the conversions that you used in part C**. They need to be accurate. You'll also need to calculate the Hubble Time, $t_H = 1/H_0$, and the Hubble Distance, $D_H = c/H_0$. So find a spot at the top to pre-compute them. Check that you are using the constants and conversions correctly using the color code. Blue cells should have equations that refer only to other cells. The black cells have all the input information needed.

F) The distance factors involve the integral shown in Eqn. 6. To compute the integral numerically, we'll find the area under the function, $f(t)=c/a(t)$ where $f(t)$ is plotted on the y-axis. A schematic of this is shown in Figure 2.



The area of the shaded trapezoid above is

$$Area = (t_2 - t_1)\left[\frac{f(t_1) + f(t_2)}{2}\right]$$

Figure 2: Schematic used to describe numerical integration. Note that the y-axis of the plot is f(t).

Column G in the spreadsheet below is the area of each trapezoid. The time difference comes from Column D and the scale factors come from Column B. Compute column G.

To integrate Eqn. 6, we need to add up the trapezoidal areas in column G between our integration limits. This is where it gets tricky because we don't want to start at the beginning of the universe, but rather at the lower integration limit, which is the current time, t_0. We want D_c to be zero at the current time: **put a zero in the column H cell where t_e = 0**. This means that light emitted right now is at zero distance from us. Now we'll add the trapezoids above to find the distance travelled by light emitted in the past. It's simplest if you have an equation like H9 = H10+G9 in your spreadsheet. Make sure column H matches the example spreadsheet below.

G) Next we need to tackle the curvature which depends on Ω_0. We will need to use some IF() statements in Excel to compute κ via Eqn. 4. These work by assigning the cell to either the first or second value based on whether the logical test is true or false: **cell value=IF(**logical_test, value_if_true, value_if_false**).** In our case, we're going to determine whether κ is +1, 0, or -1 based on the value of Ω_0.

κ = IF(Ω_0 =1, 0, IF(Ω_0 >1, 1, -1))

Look at this logic closely because one IF statement can only decide between two choices. We need to nest two of them to decide between 3 choices. In the spreadsheet below, N2 is set using the Excel equation:

```
=IF(N1=1,0,IF(N1>1,1,-1))
```

H) Curvature also has a characteristic radius of curvature, R_0. Go ahead and compute it in cell N3. Don't worry about the fact that you need to divide by zero when κ=0. For a flat universe, the curvature is infinite. This is probably the first time that #DIV/0! is the right answer.

I) You'll need more IF statements to compute $S_k(D_c)$ in column J. Take a look at Eqn. 3. The logic is:

S_k = IF($\kappa=0$, D_c, IF($\kappa=+1$, $R_0\sin(D_c/R_0)$, $R_0\sinh(D_c/R_0)$))

Debug the curvature terms, check that the $S_k = D_c$ when Ω_0=1. Check positive curvature: set Ω_0= 1.05 and check that S_k = 8275.9522 Mpc in the first bin, where log(a) = -6. Then set Ω_0=0.95 and check that S_k=8845.4898 Mpc in the first bin, where log(a) = -6.

J) Finally, you'll need to compute the angular diameter distance and the luminosity distance normalized by the Hubble Distance. See Eqns. 8 and 10. The Hubble Distance, $D_H = c/H_0$, should be pre-computed at the top of your spreadsheet. The last column is the distance modulus. It is computed from Eqn. 11. Figure 3 shows how the distance factors depend on redshift. When columns L, M, and N are debugged, you're done. Congratulations!

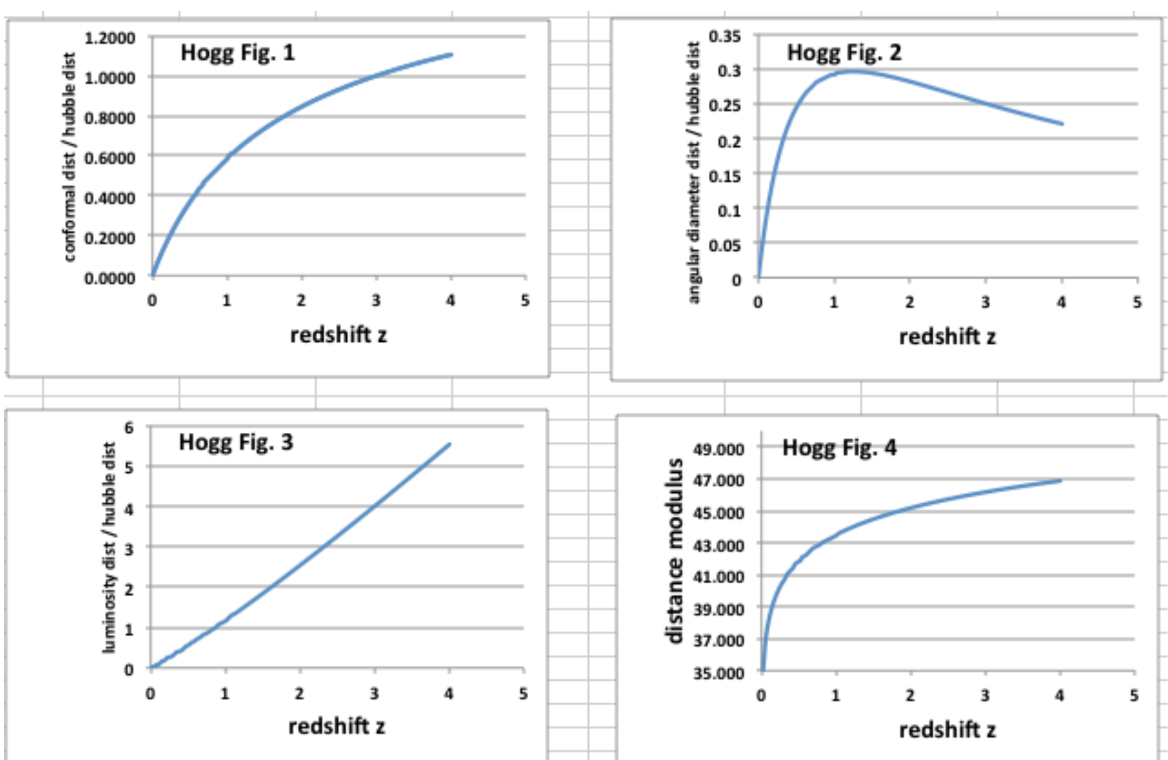


Figure 3: Plots of measurable parameters. These are recreations of Figures in reference [2] by David Hogg.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Matter Only			Ho (km/s/Mpc) =	70	c (km/s) =	2.99792E+05	D_H (Mpc) =	4.28E+03	Ω_0=	1			
2						seconds/year=	3.15581E+07	t_H (sec) =	4.41E+17	κ =	0.00E+00			
3						Mpc/km=	3.24078E-20			R_0=	#DIV/0!			
4					time since									
5			$f(a) = 2/3(a^{3/2} - 1)$	emission time	Big Bang									
6	log(a)	a	Ho(te-to)	te (sec)	age (years)	z(t)	trap Area (Mpc)	D_c (Mpc)	D_c/D_H	S_k (Mpc)	S_k/D_H	D_A/D_H	D_L/D_H	DM
7	-6	0.000001	-0.666666666	-2.93874E+17	0.000E+00	999999	0.0992	8557.8785	1.9982	8557.8785	1.9982	1.99822E-06	1998220.7	74.662
8	-5.99	1.023E-06	-0.666666666	-2.93874E+17	3.273E-01	977236.221	0.1003	8557.7793	1.9982	8557.7793	1.9982	2.04474E-06	1952713.009	74.612
9	-5.98	1.047E-06	-0.666666666	-2.93874E+17	6.660E-01	954991.586	0.1015	8557.6790	1.9982	8557.6790	1.9982	2.09235E-06	1908241.459	74.562
10	-5.97	1.072E-06	-0.666666666	-2.93874E+17	1.017E+00	933253.3008	0.1027	8557.5775	1.9982	8557.5775	1.9982	2.14106E-06	1864782.462	74.512
11	-5.96	1.096E-06	-0.666666666	-2.93874E+17	1.380E+00	912009.8394	0.1039	8557.4748	1.9981	8557.4748	1.9981	2.1909E-06	1822312.965	74.462
12	-5.95	1.122E-06	-0.666666666	-2.93874E+17	1.755E+00	891249.9381	0.1051	8557.3709	1.9981	8557.3709	1.9981	2.24191E-06	1780810.442	74.412
592	-0.15	0.7079458	-0.269558571	-1.18825E+17	5.547E+09	0.412537545	83.4620	1358.6798	0.3172	1358.6798	0.3172	0.224592087	0.448120128	41.416
593	-0.14	0.724436	-0.255603332	-1.12673E+17	5.742E+09	0.380384265	84.4284	1275.2178	0.2978	1275.2178	0.2978	0.215705741	0.411018816	41.228
594	-0.13	0.7413102	-0.241157676	-1.06305E+17	5.944E+09	0.348962883	85.4061	1190.7894	0.2780	1190.7894	0.2780	0.206116275	0.375069963	41.029
595	-0.12	0.7585776	-0.226204368	-9.97135E+16	6.153E+09	0.318256739	86.3950	1105.3833	0.2581	1105.3833	0.2581	0.195789879	0.340243813	40.818
596	-0.11	0.7762471	-0.210725568	-9.28903E+16	6.369E+09	0.288249552	87.3954	1018.9883	0.2379	1018.9883	0.2379	0.184691338	0.306511326	40.591
597	-0.1	0.7943282	-0.19470281	-8.58272E+16	6.593E+09	0.258925412	88.4074	931.5929	0.2175	931.5929	0.2175	0.172783984	0.273844161	40.346
598	-0.09	0.8128305	-0.178116978	-7.85160E+16	6.824E+09	0.230268771	89.4311	843.1854	0.1969	843.1854	0.1969	0.160029641	0.242214663	40.080
599	-0.08	0.8317638	-0.160948283	-7.09479E+16	7.064E+09	0.202264435	=(D600-D599)*0.5*($G$1/B600+$G$1/B599)*$G$3				0.1760	0.14638856	0.211595847	39.786
600	-0.07	0.851138	-0.143176244	-6.31137E+16	7.312E+09	0.174897555	91.5142	663.2876	0.1549	663.2876	0.1549	0.131819366	0.181961379	39.459
601	-0.06	0.8709636	-0.124779656	-5.50043E+16	7.569E+09	0.148153621	92.5739	571.7734	0.1335	571.7734	0.1335	0.11627899	0.153285563	39.086
602	-0.05	0.8912509	-0.105736572	-4.66099E+16	7.835E+09	0.122018454	93.6459	479.1994	0.1119	479.1994	0.1119	0.099722607	0.125543324	38.653
603	-0.04	0.9120108	-0.086024273	-3.79205E+16	8.111E+09	0.096478196	94.7303	385.5535	0.0900	385.5535	0.0900	0.082103567	0.098710199	38.130
604	-0.03	0.9332543	-0.065619242	-2.89257E+16	8.396E+09	0.071519305	95.8272	290.8233	0.0679	290.8233	0.0679	0.063373326	0.072762314	37.468
605	-0.02	0.9549926	-0.044497133	-1.96148E+16	8.691E+09	0.047128548	96.9368	194.9961	0.0455	194.9961	0.0455	0.043481373	0.047676377	36.550
606	-0.01	0.9772372	-0.022632747	-9.97678E+15	8.996E+09	0.023292992	98.0593	98.0593	0.0229	98.0593	0.0229	0.022375155	0.023429664	35.007
607	**0**	**1**	**0**	0.00000E+00	**9.312E+09**	**0**	**0.0000**	**0**	**0.0000**	**0.0000**	**0.0000**	**0**	**0**	
608	0.01	1.023293	0.023428111	1.03274E+16	9.640E+09	-0.022762779								
609	0.02	1.0471285	0.047679537	2.10177E+16	9.978E+09	-0.045007414								
610	0.03	1.0715193	0.07278321	3.20837E+16	1.033E+10	-0.066745699								
611	0.04	1.0964782	0.098769081	4.35385E+16	1.069E+10	-0.087989161								
612	0.05	1.1220185	0.125668152	5.53960E+16	1.107E+10	-0.108749062								
613	0.06	1.1481536	0.153512514	6.76701E+16	1.146E+10	-0.12903641								
614	0.07	1.1748976	0.182335387	8.03755E+16	1.186E+10	-0.148861962								
615	0.08	1.2022644	0.212171159	9.35275E+16	1.228E+10	-0.168236229								
616	0.09	1.2302688	0.243055424	1.07142E+17	1.271E+10	-0.187169484								
617	0.1	1.2589254	0.27502503	1.21234E+17	1.315E+10	-0.205671765								

Figure 4: Derived observable quantities in the Matter Only universe.

Report: Recreate Figure 1 and the plots in Figure 3 in your report. Write a few paragraphs explaining the implications of the Matter Only universe: How old is this universe? How far away is the edge of visibility? This is called the horizon distance and is defined by D_c at the time of the Big Bang. Light emitted beyond this distance has not reached planet Earth. Does this universe continue to expand forever or does it start to contract at some time in the future? According to Eqn. 7, do galaxies appear smaller ($\Delta\Omega$) at greater redshifts? Explain carefully using your plots. According to Eqn. 9, does the measured brightness go down for objects at greater redshifts?

Part 2 – General Solution to the Friedman Equation

Barbara Ryden derives the Friedman Equation, the fluid equation, and the equation of state in chapter 4. Together, they are solved in chapters 5 & 6. We will concentrate here on the general solution: (Ryden Eqn. 6.8),

$$\int_1^a \frac{da'}{\sqrt{\frac{\Omega_{r,0}}{a'^2} + \frac{\Omega_{m,0}}{a'} + \Omega_{\Lambda,0} a'^2 + (1 - \Omega_0)}} = H_0 \int_{t_0}^{t_e} dt' \qquad (13)$$

where the history of the universe is embodied in the time, t_e, and the expansion scale factor, a that is governed by the Ω parameters measured at their current epoch. This equation includes the radiation, matter, and dark energy density as well as the resulting curvature term (1-Ω_o). The right-hand side is easily integrated giving the familiar $H_0 \int_{t_0}^{t} dt' = H_0(t_e - t_0)$. The left-hand side is more complicated and has no nice analytical solution. Furthermore, it can't be inverted into the form $a(t) = f(t)$ like the Matter Only Universe. Be sure that you understand the derivation of Eqn. 13 and what it means. Once you trust the physics behind the equation, you can use it to compute interesting facts about the universe.

General Instructions: Start with the same log(a) array as you did previously. Compute a, and z as before. Next numerically integrate the left-hand-side of Eqn. 13 and set it equal to $H_0(t_e$-$t_o)$. You can use any integration method of your choice or an ODE solver. Just like before, you'll have to be careful with the integration limits. Set a=1 when t_e=t_o and then integrate backward to t_e in the past. Check that you get the results shown in column F of Figure 5 for the Ryden's Benchmark cosmology. Recreate Figure 6 below.

When the general solution for $a(t)$ is done, use it to compute the distance factors from Part 1. This shouldn't require you to re-code the distance factors. To avoid problems you want to use the code from before because it's already debugged.

Excel Instructions:

A) Open a new sheet by clicking on a new tab at the bottom of your Excel sheet. Copy log(a) into it from the previous sheet. Then compute a and z from log(a). Setup the parameters needed for the benchmark cosmology. Be careful that you set

$\Omega_0 = \Omega_{m,0}+\Omega_{r,o}+\Omega_{\Lambda,0}$. Now we need to compute the integral in Eqn. 13. The integrand is

$$f(a') = \frac{1}{\sqrt{\frac{\Omega_{r,0}}{a'^2}+\frac{\Omega_{m,0}}{a'}+\Omega_{\Lambda,0}a'^2+(1-\Omega_0)}}. \tag{14}$$

We compute the area under this function as we did before by finding the area in a series of trapezoids. Compute Column F in Figure 5 being careful with the integration limits as before. Set $H_0(t_e\text{-}t_o)$=0 at t_e=t_o then add the area of the trapezoids upward in the table from this point. Check that your results look like Figure 6.

	A	B	C	D	E	F
1	**General Solution to Friedman Equation**				**Parameters**	
2					**Omega_M**	0.3
3					**Omega_r**	8.40E-05
4					**Omega_Lambd**	0.7
5					**Omega_tot**	1.000084
6					**Ho**	70
7						
8						
9						
10	**log(a)**	**a**	**z**	**f(a)**	**trap. area**	**Ho(te-to)**
11	-6	0.000001	999999	1.09E-04	2.57E-12	-9.64E-01
12	-5.99	1.0233E-06	977236.221	1.11E-04	2.69E-12	-9.64E-01
13	-5.98	1.0471E-06	954991.586	1.14E-04	2.81E-12	-9.64E-01
14	-5.97	1.0715E-06	933253.301	1.17E-04	2.95E-12	-9.64E-01
15	-5.96	1.0965E-06	912009.839	1.19E-04	3.09E-12	-9.64E-01
16	-5.95	1.122E-06	891249.938	1.22E-04	3.23E-12	-9.64E-01
597	-0.15	0.70794578	0.41253754	1.14E+00	1.87E-02	-3.14E-01
598	-0.14	0.72443596	0.38038426	1.13E+00	1.90E-02	-2.95E-01
599	-0.13	0.74131024	0.34896288	1.13E+00	1.94E-02	-2.76E-01
600	-0.12	0.75857758	0.31825674	1.12E+00	1.97E-02	-2.57E-01
601	-0.11	0.77624712	0.28824955	1.11E+00	2.00E-02	-2.37E-01
602	-0.1	0.79432823	0.25892541	1.10E+00	2.04E-02	-2.17E-01
603	-0.09	0.81283052	0.23026877	1.10E+00	2.07E-02	-1.96E-01
604	-0.08	0.83176377	0.20226443	1.09E+00	=(B605-B604)*0.5*(D605+D604)	
605	-0.07	0.85113804	0.17489755	1.08E+00	2.13E-02	-1.55E-01
606	-0.06	0.87096359	0.14815362	1.07E+00	2.16E-02	-1.34E-01
607	-0.05	0.89125094	0.12201845	1.06E+00	2.19E-02	-1.12E-01
608	-0.04	0.91201084	0.0964782	1.05E+00	2.21E-02	-9.01E-02
609	-0.03	0.9332543	0.07151931	1.04E+00	2.24E-02	-6.80E-02
610	-0.02	0.95499259	0.04712855	1.02E+00	2.27E-02	-4.56E-02
611	-0.01	0.97723722	0.02329299	1.01E+00	2.29E-02	-2.29E-02
612	**0**	**1**	**0**	1.00E+00	**2.31E-02**	**0.00E+00**
613	0.01	1.02329299	-0.0227628	9.87E-01	2.34E-02	2.31E-02
614	0.02	1.04712855	-0.0450074	9.74E-01	2.36E-02	4.65E-02
615	0.03	1.07151931	-0.0667457	9.61E-01	2.38E-02	7.01E-02
616	0.04	1.0964782	-0.0879892	9.47E-01	2.40E-02	9.39E-02
617	0.05	1.12201845	-0.1087491	9.33E-01	2.42E-02	1.18E-01
618	0.06	1.14815362	-0.1290364	9.19E-01	2.44E-02	1.42E-01
619	0.07	1.17489755	-0.148862	9.05E-01	2.46E-02	1.67E-01
620	0.08	1.20226443	-0.1682362	8.90E-01	2.47E-02	1.91E-01
621	0.09	1.23026877	-0.1871695	8.76E-01	2.49E-02	2.16E-01
622	0.1	1.25892541	-0.2056718	8.61E-01	2.50E-02	2.41E-01
623	0.11	1.28824955	-0.2237529	8.47E-01	2.52E-02	2.66E-01

Figure 5: Spreadsheet showing the general solution to the Friedman Equation for the Benchmark Model.

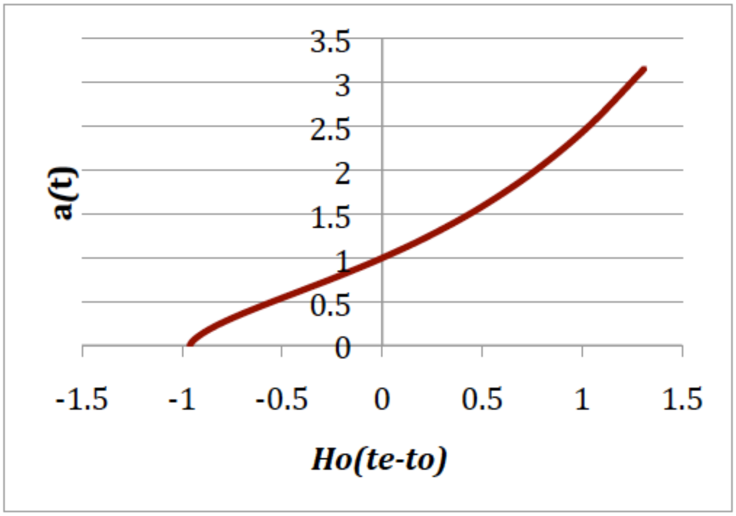


Figure 6: Scale factor as a function of time for the Benchmark Model.

B) Create another "Arbitrary" spreadsheet by copying the Matter Only spreadsheet into a new spreadsheet and relabeling the *$H_0(t_e-t_0)$* column as shown in Figure 7. Plots don't copy correctly so don't bring them along. The numbers in the black and blue columns can be copied without changing anything! Let's hook the new spreadsheet (red columns) up to the General Solution. Set the *$H_0(t_e-t_0)$* column equal to the general solution from your previous spreadsheet as shown below. "General Solution to Friedman" is the name of the spreadsheet in Figure 5. **Set *H_0* and Ω_0 to the value in the other spreadsheet as well.**

DON'T CHANGE ANY EQUATIONS! THEY SHOULD ALREADY WORK! In this part we're just hooking up the general solution to the computation of the observables. When you change the parameters in the other spreadsheet, this spreadsheet will update with the answers. Check this out by setting the omega parameters to the Matter Only case. Your Arbitrary spreadsheet should match your Matter Only spreadsheet from part 1 of the project.

	A	B	C	D	E	F
1	Arbitrary			Ho (km/s/Mpc) =	70	c (km
2						seconds/y
3						Mpc,
4					time since	
5			General Solution	emission time	Big Bang	
6	log(a)	a	Ho(te-to)	te (sec)	age (years)	
7	-6	0.000001	-9.64E-01	-4.24821E+17	0.000E+00	9
8	-5.99	1.0233E-06	-9.64E-01	-4.24821E+17	3.585E-02	97723
9	-5.98	1.0471E-06	-9.64E-01	-4.24821E+17	7.339E-02	95499
10	-5.97	1.0715E-06	='General solution to Friedman'!F14		1.127E-01	933253
11	-5.96	1.0965E-06	-9.64E-01	-4.24821E+17	1.538E-01	912009
12	-5.95	1.122E-06	-9.64E-01	-4.24821E+17	1.969E-01	891249

Figure 7: Computation for an arbitrary cosmology. Notice how column C is taken from the spreadsheet shown in Figure 5.

Report: The following can be done without modifying the columns of your spreadsheet. From here on out you should just be changing the parameters in the spreadsheet with the General Solution to the Friedman Equation.

A. Benchmark Universe: Recreate Figure 6 in your report. Verify that you compute the same age of the universe that Ryden computes in Table 6.2.
B. Now we're ready to explore several different cosmologies. Make a table with the following columns: name of cosmology, H_0, $\Omega_{r,0}$, $\Omega_{m,0}$, $\Omega_{\Lambda,0}$, Ω_0, κ, R_0, age of the universe, horizon distance. Fill it in for the Benchmark cosmology.
C. Fill in the table for the Λ-Only and Matter Only universes. The Matter Only universe is the universe from Part 1. Check that your new Arbitrary spreadsheet produces exactly the same answers! If not, you have introduced a bug in the new spreadsheets.
D. Another interesting universe is the Low Density universe. With so much empty space in the universe, lets investigate a universe that contains just baryonic matter: $\Omega_o = \Omega_{m,o} = 0.05$. What is the curvature of this universe? Add it to your table of universes. Why is this universe interesting? In the next few weeks we will find out that this universe is not the one that we actually live in even though it has all the matter that we've ever studied in the lab. We are going to look at observations and see that they differ from this universe.
E. Where were you in 1998? Until 1998 it was believed that $\Omega_{\Lambda,o}$=0, and astrophysicists focused their research on measuring Ω_o to determine the curvature of the universe. The scale factor $a(t)$ was so poorly known that the uncertainty in the age of the universe was about 50%, i.e. somewhere between 5 and 20 billion years. Today, the age of the universe is known to be 13.75 ± 0.11 billion years. This precision is better than 1% and represents a giant leap in knowledge about the origin of the universe. In just the past 12 years, the Ω parameters have been measured with extreme precision. The latest parameters can be found in Table 8 on page 39 of the WMAP paper published in January of 2010 [3]. (There is a link to the online version of the paper in the references.) Use the parameters in the WMAP+BAO+H_0 column. WMAP did not measure $\Omega_{r,0}$ which was measured by the earlier COBE satellite mission. Please use the value $\Omega_{r,0}$ = 8.40E-5 for this parameter. Add the WMAP cosmology to your table. How well does your age agree with the age in the paper?
F. Ryden's Benchmark model serves the important purpose of computing the expansion history pretty well in light of the rapid changes in the field. It is impossible to produce new editions of the book every time a paper is published. With this code, you can always produce the current age and distances as the density parameters become available. Please calculate on the percent difference in age and size between the Benchmark and WMAP cosmologies. Can you expect the computations in the textbook to be valid to a few percent?
G. Recreate the top panel in Ryden Fig. 6.6 for your cosmologies. To get all the curves on one plot you'll need to copy the redshift and D_c/D_H columns to a new spreadsheet and create the plot in the new sheet. Which curves are similar to the WMAP curves at high redshift? At low redshift?
H. Please include the table and plot in your report. Write explanations as needed to answer the various questions.

Part 3 – Ruling out the Low Density and Matter Only universes

Ryden's Figure 7.5 shows how data are used to constrain cosmological parameters. Many new type Ia supernovae have been discovered since 1999. The most recent presentation of these data are in Figure 9 on page 19 in reference [4].

A. Explain the differences between the Ryden's Figure 7.5 and Figure 9 in the new supernova paper.
B. Use all 5 cosmologies explored in Part 2 to create theory curves for the upper panel in Figure 9. (*This problem shouldn't require any additional coding. You can copy and paste results from your spreadsheest into a new spreadsheet for plotting.*)
C. Subtract the other theory predictions from the WMAP prediction to recreate the lower panel in Figure 9. The difference plot is called a residual. It is easier to compare data sets to theory predictions in this form since the differences are maximized.
D. Theoretical predictions exist for many things that do not actually exist. The data tell us what actually exists. Which of the theories we've explored are NOT consistent with this data set? How do you feel about saying that the data rule out these possibilities?

Part 4 – Measuring the size of distant galaxies

Go to the Galaxy Zoo Hubble project [5]: http://www.galaxyzoo.org/. Classify 50 galaxies and as you do so, save a variety of galaxies including galaxies with interesting structure to your album. When you're done, go to "my galaxies", select a galaxy and then click on "more information".

A. Choose a galaxy with a z>0.5 and compute its size. You'll need to click on "Show Scale" to get the angular scale. How big is the galaxy at the time it emitted the light captured by the telescope? (Please use the WMAP universe in from Part 2 to answer the question. You can interpolate between the redshift values in the table to find accurate distance factors.)
B. How does it compare to the Milky Way galaxy? How does it compare to other galaxies in the Local Group of galaxies?
C. Include a screen shot of the galaxy and it's information in your report.

References:

1. B. Ryden, *Introduction to Cosmology*, Addison Wesley, (2004).
2. D.W. Hogg, *Distance measures in cosmology*, (2000), http://arxiv.org/abs/astroph/9905116.
3. WMAP collaboration, *Seven-Year Wilkinson Microwave Anisotropy Probe Observations: Sky Maps, Systematic Errors, and Basic Results*, Jarosik, et.al., (2011) ApJS, **192**, 14; http://arxiv.org/abs/1001.4744.

4. R. Amanullah, et al. (Supernova Cosmology Project), *Spectra and Light Curves of Six Type Ia Supernovae at* $0.511 < z < 1.12$ *and the Union2 Compilation*, accepted for publication in Astrophysical Journal (2010). http://arxiv.org/abs/1004.1711 .
5. C.J. Lintott, et al., *Galaxy Zoo: Morphologies derived from visual inspection of galaxies from the Sloan Digital Sky Survey*, Mon. Not. Roy. Astron. Soc. 389:1179, (2008), www.galazyzoo.org, www.sdss.org.